On topological derivative for contact problem in elasticity

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ON TOPOLOGICAL DERIVATIVE FOR CONTACT PROBLEM IN ELASTICITY

S.M. GIUSTI*, J. SOKOŁOWSKI†, AND J. STEBEL‡

Abstract. In the paper the general method for shape-topology sensitivity analysis of contact problems is proposed. The method uses the domain decomposition method combined with the specific properties of minimizers for the energy functional. The method is applied to the static problem of an elastic body in frictionless contact with an rigid foundation. The contact model allows a finite interpenetration of the bodies on the contact region. This interpenetration is modeled by means of a scalar function that depends on the normal component of the displacement field on the potential contact zone. We present the asymptotic behavior of the energy shape functional when a spheroidal void is introduced in an arbitrary point of the elastic body. For the asymptotic analysis, we use the domain decomposition technique and the associated Steklov-Poincaré pseudodifferential operator. The differentiability of the energy with respect to the non-smooth perturbation is established. A closed form for the topological derivative is also presented.

Key words. Topological derivative, static frictionless contact problem, asymptotic analysis, domain decomposition, Steklov-Poincaré operator

AMS subject classifications. 41A60, 49J52, 49Q10, 35J50, 35Q93

1. Introduction. Topological asymptotic analysis allows us to obtain an asymptotic expansion of a given shape functional when a geometrical domain is singularly perturbed by the insertion of holes, inclusions, source-terms or even cracks. The main concept arising from this analysis is the topological derivative. This derivative can be seen as a first order correction of the unperturbed shape functional to approximate the perturbed shape functional. The topological derivative was rigorously introduced by Sokołowski & Żochowski 1999 [24]. Since then, this concept has proved to be extremely useful in the treatment of a wide range of problems, see for instance, [3, 12, 11, 9, 10, 23, 17]. Concerning the theoretical development of the topological asymptotic analysis, the reader may refer e.g. to the papers [19, 25, 5, 14, 7].

Classically, contact problems are modeled by means of a non-penetration condition between an elastic body and a rigid obstacle or foundation. This is known as *unilateral contact condition* and is modeled by using variational inequalities. A less restrictive boundary condition on the contact region is obtained by considering the *normal compliance model*. In this kind of models, some small interpenetration between the contacting bodies is allowed, and the boundary forces are given as a function of the interpenetration. However, such models allow an arbitrary large interpenetration of the bodies in contact, which is physically not very realistic. Recently, a new class of model has been presented in [6], by using a still less restrictive boundary condition that allows a *finite interpenetration* of the bodies. In such a model, the finite interpenetration is modeled by means of a function that depends on the normal component of the displacement field to the boundary on the potential contact region.

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Clearly, this is a nonlinear boundary condition for the contact problem, leading to a new class of variational inequalities.

The shape and topological asymptotic analysis for contact problems has been studied in [26, 8, 15, 4]. In these works, the differentiability of the energy functional with respect to a singular perturbation has been developed for the usual boundary conditions in contact problems. Due to the nonlinear condition over the contact zone, the boundary value problem becomes nonsmooth. Therefore, nonsmooth analysis is necessary since the shape differentiability of solutions to contact problems is obtained only in the framework of Hadamard differentiability of metric projections onto polyhedric sets in the appropriate Sobolev spaces.

In this work we present the asymptotic behavior of the energy shape functional when a spheroidal void is introduced in an arbitrary point of the elastic body. We consider the energy shape functional associated to the frictionless contact problem allowing a finite interpenetration between an elastic body and a rigid foundation, developed in [6]. For the asymptotic analysis, we use the domain decomposition technique and the associated Steklov-Poincaré pseudodifferential operator. The differentiability of the energy of this new class of variational problem, with respect to the non-smooth perturbation, is established. A closed form for the topological derivative in the three-dimensional space is also presented.

The paper is organized as follows. The problem formulation associated with contact problem, without friction and finite interpenetration, is presented in Section 2. The topological asymptotic analysis with respect to the nucleation of spherical holes (voids) in 3D is developed with all details in Section 3. Here, a closed form of the topological derivatives associated with the energy shape functional is presented. The paper ends with some concluding remarks in Section 4.

2. Static contact model for finite interpenetration. We consider the problem of an elastic body having contact with a rigid foundation. The domain of the body, denoted by $\Omega \subset \mathbb{R}^3$, is assumed to be bounded and have Lipschitz boundary $\partial\Omega$ consisting of three mutually disjoint parts with positive measures Γ_D, Γ_N and Γ_C, where different boundary conditions are prescribed. On the boundary Γ_D we prescribe Dirichlet boundary conditions (displacement), on Γ_N Neumann boundary conditions (traction) and, finally, on Γ_C the contact condition with the rigid foundation that admits an interpenetration, see Figure 1. For the contact model on Γ_C, we consider only a normal compliance law of the type

$$\sigma_n(u) = -p(u_n - g)\,, \tag{2.1}$$

where $u_n := u \cdot n$ denotes the normal component of the displacement field u, n is the unit outward normal vector to the boundary $\partial\Omega$ and g the gap on the potential contact zone. Moreover, in (2.1), $\sigma_n(u)$ represents the normal component to the boundary of the stress tensor $\sigma(u)$, i.e. $\sigma_n(u) = \sigma(u)n \cdot n$. The Cauchy stress tensor $\sigma(u)$ is defined as:

$$\sigma(u) := \mathbb{C}\varepsilon(u)\,, \tag{2.2}$$

where $\varepsilon(u)$ is the symmetric part of the gradient of the displacement field u, i.e.

$$\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^\top)\,, \tag{2.3}$$

and $\mathbb{C}$ denotes the four-order elastic tensor. For an isotropic elastic body, this tensor is given by:

$$\mathbb{C} = 2\mu\mathbb{I} + \lambda(\mathrm{I}\otimes\mathrm{I}), \tag{2.4}$$

with μ and λ denoting the Lame's coefficients. In the above expression, we use $\mathbb{I}$ and I to denote, respectively, the identities of fourth and second order. In terms of the engineering constant E (Young's modulus) and υ (Poisson's ratio) the above constitutive response can be written as:

$$\mathbb{C} = \frac{E}{1-\upsilon^2}\left[(1-\upsilon)\mathbb{I} + \upsilon(\mathrm{I}\otimes\mathrm{I})\right]. \tag{2.5}$$

The function $p : \mathbb{R} \to \overline{\mathbb{R}_+} = [0, +\infty]$ in (2.1) is used to model the interpenetration condition between the body and the foundation. This function p is monotone with the following properties:

$$\left\{\begin{array}{rcll} p(y) & = & 0 & \text{for } y \le \alpha, \text{ with } \alpha \text{ constant} \\ \lim\limits_{y\to\beta-} p(y) & = & +\infty & \text{for } y > \alpha, \text{ with } \beta \text{ constant and } \beta > \alpha \\ p(y) & = & +\infty & \text{for } y \ge \beta \end{array}\right. . \tag{2.6}$$

The parameter α indicates the initial contact and the value of β describes a limit such that no further interpenetration is possible.

The strong form of the equilibrium equation under this contact condition is given by: find the displacement field $u : \Omega \to \mathbb{R}^3$ such that

$$\left\{\begin{array}{rcll} -\operatorname{div}\sigma(u) & = & 0 & \text{in} \quad \Omega \\ u & = & \overline{u} & \text{on} \quad \Gamma_D \\ \sigma(u)n & = & \overline{t} & \text{on} \quad \Gamma_N \\ \sigma_n(u) & = & -p(u_n - g) & \text{on} \quad \Gamma_C \\ \sigma_\tau(u) & = & 0 & \text{on} \quad \Gamma_C \end{array}\right. . \tag{2.7}$$

The last condition in (2.7) indicates that the contact is without friction, where $\sigma_\tau(u) = \sigma(u)n - \sigma_n(u)n$ denotes the tangential component of the stress tensor $\sigma(u)$.

We assume that the stress operator σ is bounded and positive definite, i.e. there exist two constants $\underline{\sigma}, \overline{\sigma} > 0$ such that:

$$|\sigma| \le \overline{\sigma}, \quad \forall \phi \in \mathbb{R}^{3\times 3} : \sigma(\phi)\cdot\phi \ge \underline{\sigma}|\phi|^2, \tag{2.8}$$

and the data satisfy:

$$g \in H^{1/2}(\Gamma_C), \quad \overline{u} \in H^1(\Omega;\mathbb{R}^3), \quad \overline{u}_n|_{\Gamma_C} = g \text{ and } \overline{t} \in (H^{1/2}(\Gamma_N;\mathbb{R}^3))^*. \tag{2.9}$$

The weak formulation of the problem stated in (2.7) is given by the following variational equation: find $u \in \mathcal{U}$ with $(u_n - g) \in \operatorname{dom}(p)$, such that:

$$\langle\sigma(u), \varepsilon(v) - \varepsilon(u)\rangle_\Omega + \langle p(u_n - g), v_n - u_n\rangle_{\Gamma_C} = \langle\overline{t}, v - u\rangle_{\Gamma_N} \quad \forall v \in \mathcal{U}, \tag{2.10}$$

where the set of admissible functions $\mathcal{U}$ is given by:

$$\mathcal{U} := \{\varphi \in H^1(\Omega;\mathbb{R}^3) : \varphi = \overline{u} \quad \text{on} \quad \Gamma_D\}, \tag{2.11}$$

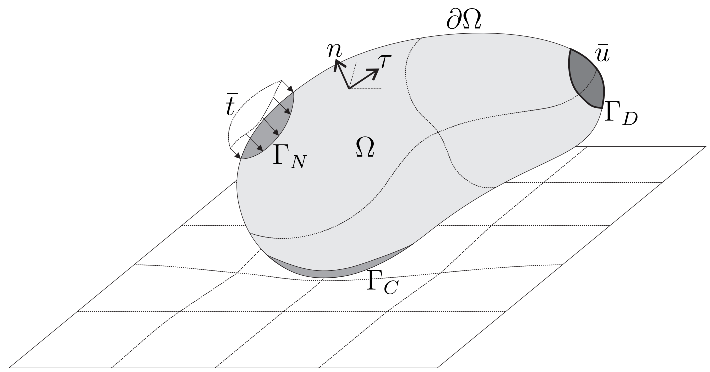


FIG. 1. *Contact problem.*

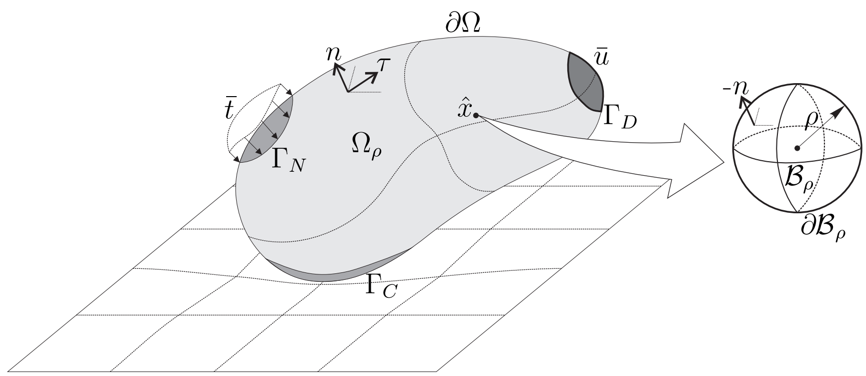


FIG. 2. *Perturbed contact problem.*

and the domain of definition of the function p, namely $\mathrm{dom}(p)$, is:

$$\mathrm{dom}(p) := \left\{ \varphi \in L^1(\Gamma_C) \,:\, p(\varphi) \in L^1(\Gamma_C), \exists C > 0 \,:\, \int_{\Gamma_C} p(\varphi) v \leq C \|v\|_{H^{1/2}(\Gamma_C)} \right\}. \tag{2.12}$$

For a complete and detailed description of this model, we refer the reader to [6], where it was proved that, under the above assumptions, problem (2.10) admits a unique solution.

3. Topological asymptotic analysis. In this section we obtain an asymptotic expansion for the energy shape functional when a small spheroidal cavity of radius ρ is introduced in an arbitrary point $\hat{x}$ of the domain Ω, far enough from the potential contact region Γ_C, see Figure 2. The main term of this expansion is the topological derivative and represents a first order asymptotic correction term of a given shape functional with respect to a singular domain perturbation [24].

Let us consider a shape functional defined on the domain Ω and depending on the solution u, denoted by $\mathcal{J}_\Omega(u)$. Then, after the introduction of a singular perturbation at $\hat{x}$, we have a new domain denoted by $\Omega_\rho := \Omega \setminus \overline{\mathcal{B}_\rho}$, where $\mathcal{B}_\rho$ is a ball of radius ρ centered at $\hat{x}$, that is $\mathcal{B}_\rho := \left\{ x \in \mathbb{R}^3 \,:\, |x - \hat{x}| < \rho \right\}$, see Figure 2.

Therefore, an asymptotic expansion of the energy shape functional defined on the perturbed domain Ω_ρ, i.e. $\mathcal{J}_{\Omega_\rho}$, can be written as:

$$\mathcal{J}_{\Omega_\rho}(u_\rho) = \mathcal{J}_\Omega(u) + f(\rho)\mathcal{T}_\Omega(\hat{x}) + o(f(\rho)), \tag{3.1}$$

where $f(\rho)$ is a positive function that decreases monotonically, such that $f(\rho) \to 0$ when $\rho \to 0^+$, $\mathcal{T}_\Omega(\hat{x})$ is defined as the topological derivative of $\mathcal{J}_\Omega$ at $\hat{x}$, and u_ρ is the solution of the contact problem in the perturbed domain given by: find the displacement field $u_\rho : \Omega_\rho \to \mathbb{R}^3$ such that

$$\left\{ \begin{array}{rcllc} -\operatorname{div}\sigma(u_\rho) & = & 0 & \text{in} & \Omega_\rho \\ u_\rho & = & \overline{u} & \text{on} & \Gamma_D \\ \sigma(u_\rho)n & = & \overline{t} & \text{on} & \Gamma_N \\ \sigma_n(u_\rho) & = & -p(u_{\rho n} - g) & \text{on} & \Gamma_C \\ \sigma_\tau(u_\rho) & = & 0 & \text{on} & \Gamma_C \\ \sigma(u_\rho)n & = & 0 & \text{on} & \partial\mathcal{B}_\rho \end{array} \right. , \tag{3.2}$$

where $u_{\rho n} := u_\rho \cdot n$ is used to denote the normal component of the displacement field u_ρ on the boundary Γ_C. Note that there is no traction applied on the boundary of the hole, i.e. homogeneous Neumman boundary condition has been considered on $\partial\mathcal{B}_\rho$ for this problem.

From (3.1) we have that the classical definition of the topological derivative is given by [24]:

$$\mathcal{T}_\Omega(\hat{x}) := \lim_{\rho \to 0^+} \frac{\mathcal{J}_{\Omega_\rho}(u_\rho) - \mathcal{J}_\Omega(u)}{f(\rho)} \,. \tag{3.3}$$

In order to perform the asymptotic expansion and the evaluation of the topological derivative of problem (3.2), in this work we apply the domain decomposition method and the associated Steklov-Poincaré pseudodifferential operator.

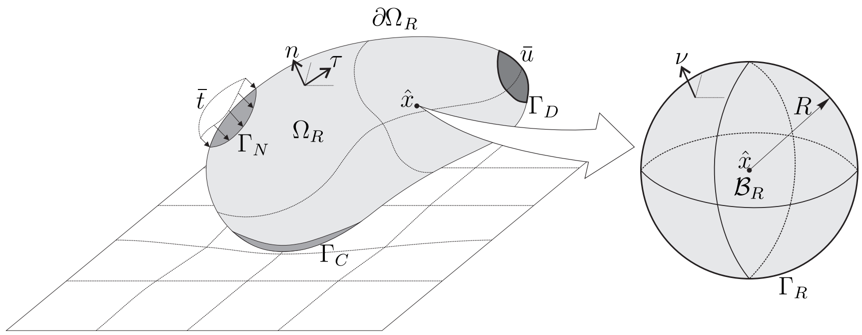
3.1. Domain decomposition. We start by decomposing the domain Ω_ρ in two parts: (i) a ball $\mathcal{B}_R$ of radius $R > \rho > 0$ centered at $\hat{x} \in \Omega$, that is $\mathcal{B}_R := \{x \in \mathbb{R}^3 : |x - \hat{x}| < R\}$, and (ii) the domain $\Omega_R := \Omega \setminus \overline{\mathcal{B}_R}$. Clearly, the domain $\mathcal{B}_R$ contains the small cavity $\mathcal{B}_\rho$ and, for this perturbed configuration, we can define the domain as $C(R,\rho) := \mathcal{B}_R \setminus \overline{\mathcal{B}_\rho}$, see Figure 3. We use Γ_R to denote the exterior boundary $\partial\mathcal{B}_R$ of the domain $C(R,\rho)$. First we consider the following linear elasticity system in $C(R,\rho)$: given $\overline{v} \in H^{1/2}(\Gamma_R;\mathbb{R}^3)$, find the displacement field $\omega_\rho : C(R,\rho) \to \mathbb{R}^3$ such that

$$\left\{ \begin{array}{rcllc} -\operatorname{div}\sigma(\omega_\rho) & = & 0 & \text{in} & C(R,\rho) \\ \omega_\rho & = & \overline{v} & \text{on} & \Gamma_R \\ \sigma(\omega_\rho)n & = & 0 & \text{on} & \partial\mathcal{B}_\rho \end{array} \right. . \tag{3.4}$$

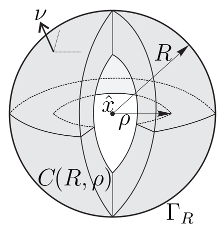
Using (3.4) we define the Steklov-Poincaré boundary operator $\mathcal{S}_\rho$ on Γ_R as:

$$\mathcal{S}_\rho \, : \, \overline{v} \in H^{1/2}(\Gamma_R;\mathbb{R}^3) \to \sigma(\omega_\rho)\nu \in H^{-1/2}(\Gamma_R;\mathbb{R}^3), \tag{3.5}$$

where ν denotes the unit normal vector to the boundary Γ_R pointing outside the ball $\mathcal{B}_R$. Next, we consider the following contact problem in Ω_R: find the displacement



(a) Domain Ω_R and $\mathcal{B}_R$.



(b) Domain $C(R,\rho)$.

FIG. 3. *Decomposition of the domain* Ω.

field $u_\rho^R : \Omega_R \to \mathbb{R}^3$, such that:

$$\left\{ \begin{array}{rcllr} -\operatorname{div}\sigma(u_\rho^R) & = & 0 & \text{in} & \Omega_R \\ u_\rho^R & = & \overline{u} & \text{on} & \Gamma_D \\ \sigma(u_\rho^R)n & = & \overline{t} & \text{on} & \Gamma_N \\ \sigma_n(u_\rho^R) & = & -p(u_{\rho n}^R - g) & \text{on} & \Gamma_C \\ \sigma_\tau(u_\rho^R) & = & 0 & \text{on} & \Gamma_C \\ \sigma(u_\rho^R)\nu & = & \mathcal{S}_\rho(u_\rho^R) & \text{on} & \Gamma_R \end{array} \right. . \tag{3.6}$$

Its variational formulation can be written as: find the displacement field $u_\rho^R \in \mathcal{U}_R$ with $p(u_{\rho n}^R - g) \in \operatorname{dom}(p)$, such that:

$$\langle \sigma(u_\rho^R), \varepsilon(v) - \varepsilon(u_\rho^R)\rangle_{\Omega_R} + \langle p(u_{\rho_n}^R - g), v_n - u_{\rho n}^R\rangle_{\Gamma_C} \\ + \langle \mathcal{S}_\rho(u_\rho^R), v - u_\rho^R\rangle_{\Gamma_R} = \langle \overline{t}, v - u_\rho^R\rangle_{\Gamma_N} \quad \forall v \in \mathcal{U}_R\,, \tag{3.7}$$

where the set of admissible functions $\mathcal{U}_R$ is given by:

$$\mathcal{U}_R := \{\varphi \in H^1(\Omega_R;\mathbb{R}^3) \,:\, \varphi = \overline{u} \quad \text{on} \quad \Gamma_D\}. \tag{3.8}$$

From (3.4) and (3.5) it follows that the solution u_ρ of (3.2) satisfies $\sigma(u_\rho)\nu = \mathcal{S}_\rho(u_\rho)$ on Γ_R. Consequently, the restriction of u_ρ to the truncated domain Ω_R coinsides with the solution u_ρ^R of (3.6) and similarly $u_\rho|_{C(R,\rho)} = \omega_\rho$, where ω_ρ is the solution to (3.4) with $\overline{v} = u_\rho|_{\Gamma_R}$.

We also observe that, by the definition of the Steklov-Poincaré boundary operator in the domain $C(R,\rho)$, the solution ω_ρ of (3.4) satisfies

$$\int_{C(R,\rho)} \sigma(\omega_\rho)\cdot\varepsilon(v) = \langle \mathcal{S}_\rho(\omega_\rho), v\rangle_{\Gamma_R} \quad \forall v \in \mathcal{U}_C\,, \tag{3.9}$$

where the set of admissible function $\mathcal{U}_C$ is given by:

$$\mathcal{U}_C := \{\varphi \in H^1(C(R,\rho);\mathbb{R}^3) \,:\, \varphi = \overline{v} \quad \text{on} \quad \Gamma_R\}. \tag{3.10}$$

For the unperturbed case ($\rho = 0$) we define the Steklov-Poincaré operator $\mathcal{S} := \mathcal{S}_0 \,:\, \overline{v} \in H^{1/2}(\Gamma_R;\mathbb{R}^3) \to \sigma(\omega)\nu \in H^{-1/2}(\Gamma_R;\mathbb{R}^3)$ associated with the problem

$$\left\{ \begin{array}{rclcl} -\operatorname{div}\sigma(\omega) & = & 0 & \text{in} & \mathcal{B}_R \\ \omega & = & \overline{v} & \text{on} & \Gamma_R \end{array} \right. . \tag{3.11}$$

Applying the domain decomposition technique to the problem (2.7) on Ω, we can rewrite (2.10) as follows:

$$\langle\sigma(u),\varepsilon(v)-\varepsilon(u)\rangle_{\Omega_R}+\langle p(u_n-g),v_n-u_n\rangle_{\Gamma_C}+\langle\mathcal{S}(u),v-u\rangle_{\Gamma_R} = \langle\bar{t},v-u\rangle_{\Gamma_N} \quad \forall v \in \mathcal{U}_R. \tag{3.12}$$

It is well known that $\mathcal{S}_\rho$ is a positive definite operator for any $\rho \geq 0$, and that the following asymptotic expansion holds:

$$\mathcal{S}_\rho = \mathcal{S} + \rho^3\mathcal{S}' + o(\rho^3)\,,\;\; \rho \to 0^+\,, \tag{3.13}$$

with a bounded linear operator $\mathcal{S}'$ [26].

3.2. Topological derivative. For the contact model studied in this work, the energy shape functional associated to the domain Ω is given by [6]:

$$\mathcal{J}_\Omega(u) := \frac{1}{2}\langle\sigma(u),\varepsilon(u)\rangle_\Omega - \langle\bar{t},u\rangle_{\Gamma_N} + \int_{\Gamma_C} P(u_n-g), \tag{3.14}$$

where u denotes the solution of the problem in the unperturbed domain, see (2.7), and the function $P(y)$ is given by:

$$P(y) := \int_{-\infty}^{y} p(z). \tag{3.15}$$

Considering the singular perturbation $\mathcal{B}_\rho$, the energy shape functional associated to the perturbed domain Ω_ρ is given by:

$$\mathcal{J}_{\Omega_\rho}(u_\rho) := \frac{1}{2}\langle\sigma(u_\rho),\varepsilon(u_\rho)\rangle_{\Omega_\rho} - \langle\bar{t},u_\rho\rangle_{\Gamma_N} + \int_{\Gamma_C} P(u_{\rho n}-g), \tag{3.16}$$

where u_ρ is the solution of the problem in the domain Ω_ρ, see (3.2).

Now, by taking into account the domain decomposition and the Steklov-Poincaré boundary operator presented above, we can define the following functional associated to the truncated domain Ω_R:

$$\mathcal{I}_{\Omega_R}(u_\rho^R) := \frac{1}{2}\langle\sigma(u_\rho^R),\varepsilon(u_\rho^R)\rangle_{\Omega_R} - \langle\bar{t},u_\rho^R\rangle_{\Gamma_N} + \int_{\Gamma_C} P(u_{\rho n}^R-g) + \frac{1}{2}\langle\mathcal{S}_\rho(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}. \tag{3.17}$$

In view of the above functional, the contact problem in the truncated domain Ω_R, given by eq.(3.6), can be written as the following optimization problem: the displacement field u_ρ^R is the unique minimizer such that

$$\mathcal{I}_{\Omega_R}(u_\rho^R) = \inf_{v\in dom(\mathcal{I}_{\Omega_R})}\{\mathcal{I}_{\Omega_R}(v)\}\,, \tag{3.18}$$

where the domain of the functional $\mathcal{I}_{\Omega_R}$ is given by:

$$\mathrm{dom}(\mathcal{I}_{\Omega_R}) := \left\{ v \in \mathcal{U}_R \,:\, P(v_n - g) \in L^1(\Gamma_C) \right\} . \tag{3.19}$$

For the optimization problem (3.18), we can establish the following equivalence

$$\mathcal{I}_{\Omega_R}(u_\rho^R) \equiv \mathcal{J}_{\Omega_\rho}(u_\rho), \tag{3.20}$$

since the minimizer in (3.18) coincides with the restriction to Ω_R of the minimizer u_ρ of the corresponding quadratic functional defined in the whole singularly perturbed domain Ω_ρ.

Proposition 1. *Let u and u_ρ be the solutions to* (2.10) *and* (3.7)*, respectively. Then*

$$u_\rho \to u \qquad \textit{strongly in} \qquad H^1(\Omega_R, \mathbb{R}^3), \;\; \textit{as } \rho \to 0^+. \tag{3.21}$$

Proof. First we show that the sequence $\{u_\rho\}$, $\rho \to 0^+$, is bounded in $H^1(\Omega_R; \mathbb{R}^3)$. Using $v := 2u_\rho - \overline{u}$ as a test function in (3.7) we obtain:

$$\begin{aligned} &\langle \sigma(u_\rho), \varepsilon(u_\rho) \rangle_{\Omega_R} + \langle p(u_{\rho n} - g), u_{\rho n} - g \rangle_{\Gamma_C} + \langle \mathcal{S}_\rho(u_\rho), u_\rho \rangle \\ &= \langle \sigma(u_\rho), \varepsilon(\overline{u}) \rangle_{\Omega_R} + \langle p(u_{\rho n} - g), \overline{u}_n - g \rangle_{\Gamma_C} + \langle \mathcal{S}_\rho(u_\rho), \overline{u} \rangle_{\Gamma_R} + \langle \bar{t}, u_\rho - \overline{u} \rangle_{\Gamma_N} . \end{aligned} \tag{3.22}$$

The terms on the right hand side can be estimated using the boundedness of σ, the expression (3.9) and the properties of the data $\overline{u}$ and $\bar{t}$ as follows:

$$\begin{aligned} \langle \sigma(u_\rho), \varepsilon(\overline{u}) \rangle_{\Omega_R} + \langle \mathcal{S}_\rho(u_\rho), \overline{u} \rangle_{\Gamma_R} &= \langle \sigma(u_\rho), \varepsilon(\overline{u}) \rangle_{\Omega_\rho} \\ &\le \overline{\sigma} \| \varepsilon(u_\rho) \|_{L^2(\Omega_\rho, \mathbb{R}^3)} \| \varepsilon(\overline{u}) \|_{L^2(\Omega_\rho, \mathbb{R}^3)}, \qquad (3.23) \\ \langle p(u_{\rho n} - g), \overline{u}_n - g \rangle_{\Gamma_C} &= 0, \qquad (3.24) \\ \langle \bar{t}, u_\rho - \overline{u} \rangle_{\Gamma_N} &\le \| \bar{t} \|_{(H^{1/2}(\Gamma_N; \mathbb{R}^3))^*} \big(\| u_\rho \|_{H^{1/2}(\Gamma_N; \mathbb{R}^3)} \\ &\qquad + \| \overline{u} \|_{H^{1/2}(\Gamma_N; \mathbb{R}^3)} \big) . \qquad (3.25) \end{aligned}$$

Using positive definiteness of σ, the expression (3.9) and the monotonicity of p we get the lower bound for the left hand side of (3.22):

$$\langle \sigma(u_\rho), \varepsilon(u_\rho) \rangle_{\Omega_R} + \langle p(u_{\rho n} - g), u_{\rho n} - g \rangle_{\Gamma_C} + \langle \mathcal{S}_\rho(u_\rho), u_\rho \rangle_{\Gamma_R} \ge \underline{\sigma} \| \varepsilon(u_\rho) \|^2_{L^2(\Omega_\rho; \mathbb{R}^3)} . \tag{3.26}$$

Combining the above estimates with (3.22) we find that there is a constant $C_1 > 0$ depending only on $\underline{\sigma}$, $\overline{\sigma}$, $\|\overline{u}\|_{H^1(\Omega; \mathbb{R}^3)}$ and $\|\bar{t}\|_{(H^{1/2}(\Gamma_N; \mathbb{R}^3))^*}$ such that

$$\| \varepsilon(u_\rho) \|^2_{L^2(\Omega_\rho; \mathbb{R}^3)} \le C_1 (\| \varepsilon(u_\rho) \|_{L^2(\Omega_\rho; \mathbb{R}^3)} + \| u_\rho \|_{H^{1/2}(\Gamma_N; \mathbb{R}^3)} + \| \overline{u} \|_{H^{1/2}(\Gamma_N; \mathbb{R}^3)}). \tag{3.27}$$

Now we use the embedding $H^1(\Omega_R; \mathbb{R}^3) \hookrightarrow H^{1/2}(\Gamma_N; \mathbb{R}^3)$, Young's inequality and Korn's inequality in $H^1(\Omega_R; \mathbb{R}^3)$ which yields:

$$\| u_\rho \|_{H^1(\Omega_R; \mathbb{R}^3)} \le C_K \| \varepsilon(u_\rho) \|_{L^2(\Omega_R; \mathbb{R}^3)} \le C_K \| \varepsilon(u_\rho) \|_{L^2(\Omega_\rho, \mathbb{R}^3)} \le C_2, \tag{3.28}$$

where $C_K > 0$ is the constant of the Korn inequality and $C_2 > 0$ depends on the same quantities as C_1.

To show strong convergence, we test (3.7) by $v := u$ and (3.12) by $v := u_\rho$. Adding the resulting equations and multiplying by -1 we obtain:

$$\langle \sigma(u_\rho) - \sigma(u), \varepsilon(u_\rho) - \varepsilon(u)\rangle_{\Omega_R} + \langle p(u_{\rho n} - g) - p(u - g), u_{\rho n} - u_n\rangle_{\Gamma_C} + \langle \mathcal{S}_\rho(u_\rho) - \mathcal{S}(u), u_\rho - u\rangle_{\Gamma_R} = 0. \quad (3.29)$$

Since $\mathcal{S}_\rho$ is a positive definite operator which admits the asymptotic expansion (3.13) and the sequence $\{u_\rho\}$ is bounded in $H^1(\Omega_R; \mathbb{R}^3)$, the last term in (3.29) satisfies:

$$\begin{aligned}\langle \mathcal{S}_\rho(u_\rho) - \mathcal{S}(u), u_\rho - u\rangle_{\Gamma_R} &= \langle \mathcal{S}_\rho(u_\rho) - \mathcal{S}_\rho(u), u_\rho - u\rangle_{\Gamma_R} + \langle \mathcal{S}_\rho(u) - \mathcal{S}(u), u_\rho - u\rangle_{\Gamma_R} \\ &\geq \langle \rho^3 \mathcal{S}'(u), u_\rho - u\rangle_{\Gamma_R} + o(\rho^3) \to 0 \quad \text{as } \rho \to 0^+. \end{aligned} \quad (3.30)$$

Using this, together with the Korn inequality and the facts that σ is positive definite and p is non-decreasing, we deduce from (3.29) that

$$\lim_{\rho \to 0^+} \|u_\rho - u\|^2_{H^1(\Omega_R;\mathbb{R}^3)} \leq 0, \quad (3.31)$$

which completes the proof. □

Proposition 2. *The functional form $\mathcal{I}_{\Omega_R}$ defined in (3.17) is differentiable at $\rho = 0^+$, for any fixed $R > \rho$ with $\rho \geq 0$, and the derivative is*

$$\mathcal{I}'_\Omega = \frac{1}{2}\int_{\Gamma_R} \mathcal{S}'(u^R)u^R = \frac{1}{2}\langle \mathcal{S}'(u^R), u^R\rangle_{\Gamma_R}, \quad (3.32)$$

where $\mathcal{S}'$ is the main term of the asymptotic expansion of the Steklov-Poincaré boundary operator $\mathcal{S}_\rho$ in the space of the Steklov-Poincaré operators, given by:

$$\mathcal{S}_\rho = \mathcal{S} + \rho^3 \mathcal{S}' + o(\rho^3). \quad (3.33)$$

Proof. The derivative of the functional form $\mathcal{I}_{\Omega_R}$ at $\rho = 0^+$ can be written as:

$$\mathcal{I}'_\Omega := \lim_{\rho \to 0^+} \frac{\mathcal{I}_{\Omega_R}(u^R_\rho) - \mathcal{I}_\Omega(u^R)}{\rho^3}. \quad (3.34)$$

Let us consider the following inequalities

$$\frac{\mathcal{I}_{\Omega_R}(u^R_\rho) - \mathcal{I}_\Omega(u^R_\rho)}{\rho^3} \leq \mathcal{I}'_\Omega \leq \frac{\mathcal{I}_{\Omega_R}(u^R) - \mathcal{I}_\Omega(u^R)}{\rho^3}. \quad (3.35)$$

Now, for the left-hand side of (3.35) we have that

$$
\begin{aligned}
\frac{\mathcal{I}_{\Omega_R}(u_\rho^R)-\mathcal{I}_\Omega(u_\rho^R)}{\rho^3} &= \frac{1}{\rho^3}\Bigg\{\frac{1}{2}\langle\sigma(u_\rho^R),\varepsilon(u_\rho^R)\rangle_{\Omega_R}-\langle\bar{t},u_\rho^R\rangle_{\Gamma_N}+\int_{\Gamma_C}P(u_{\rho_n}^R-g)\\
&\quad+\frac{1}{2}\langle\mathcal{S}_\rho(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}-\frac{1}{2}\langle\sigma(u_\rho^R),\varepsilon(u_\rho^R)\rangle_\Omega+\langle\bar{t},u_\rho^R\rangle_{\Gamma_N}\\
&\quad-\int_{\Gamma_C}P(u_{\rho_n}^R-g)\Bigg\}\\
&= \frac{1}{\rho^3}\Bigg\{\frac{1}{2}\langle\sigma(u_\rho^R),\varepsilon(u_\rho^R)\rangle_{\Omega_R}-\langle\bar{t},u_\rho^R\rangle_{\Gamma_N}+\int_{\Gamma_C}P(u_{\rho_n}^R-g)\\
&\quad+\frac{1}{2}\langle\mathcal{S}_\rho(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}-\frac{1}{2}\langle\sigma(u_\rho^R),\varepsilon(u_\rho^R)\rangle_{\Omega^R}+\langle\bar{t},u_\rho^R\rangle_{\Gamma_N}\\
&\quad-\int_{\Gamma_C}P(u_{\rho_n}^R-g)-\frac{1}{2}\langle\mathcal{S}(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}\Bigg\}\\
&= \frac{1}{2\rho^3}\langle\mathcal{S}_\rho(u_\rho^R)-\mathcal{S}(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}
\end{aligned}
\tag{3.36}
$$

Considering the asymptotic expansion of the Steklov-Poincaré operator, we have:

$$
\begin{aligned}
\frac{\mathcal{I}_{\Omega_R}(u_\rho^R)-\mathcal{I}_\Omega(u_\rho^R)}{\rho^3} &= \frac{1}{2\rho^3}\langle\mathcal{S}(u_\rho^R)+\rho^3\mathcal{S}'(u_\rho^R)+o(\rho^3)-\mathcal{S}(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}\\
&= \frac{1}{2}\langle\mathcal{S}'(u_\rho^R),u_\rho^R\rangle_{\Gamma_R}+\frac{1}{2}\langle\frac{o(\rho^3)}{\rho^3},u_\rho^R\rangle_{\Gamma_R}\,.
\end{aligned}
\tag{3.37}
$$

Using the strong convergence of u_ρ^R to u^R and the linearity of $\mathcal{S}'$ we obtain:

$$
\lim_{\rho\to 0^+}\frac{\mathcal{I}_{\Omega_R}(u_\rho^R)-\mathcal{I}_\Omega(u_\rho^R)}{\rho^3}=\frac{1}{2}\langle\mathcal{S}'(u^R),u^R\rangle_{\Gamma_R}. \tag{3.38}
$$

Now, the right-hand side of (3.35) can be written as:

$$
\begin{aligned}
\frac{\mathcal{I}_{\Omega_R}(u^R)-\mathcal{I}_\Omega(u^R)}{\rho^3} &= \frac{1}{\rho^3}\Bigg\{\frac{1}{2}\langle\sigma(u^R),\varepsilon(u^R)\rangle_{\Omega_R}-\langle\bar{t},u^R\rangle_{\Gamma_N}+\int_{\Gamma_C}P(u_n^R-g)\\
&\quad+\frac{1}{2}\langle\mathcal{S}_\rho(u^R),u^R\rangle_{\Gamma_R}-\frac{1}{2}\langle\sigma(u^R),\varepsilon(u^R)\rangle_\Omega+\langle\bar{t},u^R\rangle_{\Gamma_N}\\
&\quad-\int_{\Gamma_C}P(u_n^R-g)\Bigg\}\\
&= \frac{1}{\rho^3}\Bigg\{\frac{1}{2}\langle\sigma(u^R),\varepsilon(u^R)\rangle_{\Omega_R}-\langle\bar{t},u^R\rangle_{\Gamma_N}+\int_{\Gamma_C}P(u_n^R-g)\\
&\quad+\frac{1}{2}\langle\mathcal{S}_\rho(u^R),u^R\rangle_{\Gamma_R}-\frac{1}{2}\langle\sigma(u^R),\varepsilon(u^R)\rangle_{\Omega^R}+\langle\bar{t},u^R\rangle_{\Gamma_N}\\
&\quad-\int_{\Gamma_C}P(u_n^R-g)-\frac{1}{2}\langle\mathcal{S}(u^R),u^R\rangle_{\Gamma_R}\Bigg\}\\
&= \frac{1}{2\rho^3}\langle\mathcal{S}_\rho(u^R)-\mathcal{S}(u^R),u^R\rangle_{\Gamma_R}\,.
\end{aligned}
\tag{3.39}
$$

Considering the asymptotic expansion of the Steklov-Poincaré operator, we have:

$$\begin{aligned}\frac{\mathcal{I}_{\Omega_R}(u^R)-\mathcal{I}_\Omega(u^R)}{\rho^3} &= \frac{1}{2\rho^3}\langle \mathcal{S}(u^R)+\rho^3\mathcal{S}'(u^R)+o(\rho^3)-\mathcal{S}(u^R),u^R\rangle_{\Gamma_R}\\ &= \frac{1}{2}\langle \mathcal{S}'(u^R),u^R\rangle_{\Gamma_R}+\frac{1}{2}\langle\frac{o(\rho^3)}{\rho^3},u^R\rangle_{\Gamma_R} \qquad (3.40)\end{aligned}$$

By taking the limit of the above expression when $\rho\to 0^+$, we obtain:

$$\lim_{\rho\to 0^+}\frac{\mathcal{I}_{\Omega_R}(u^R)-\mathcal{I}_\Omega(u^R)}{\rho^3}=\frac{1}{2}\langle \mathcal{S}'(u^R),u^R\rangle_{\Gamma_R}. \qquad (3.41)$$

Finally, from expressions (3.38) and (3.41), it follows (3.32). □

Using Proposition 2, the asymptotic expansion of the functional $\mathcal{I}_{\Omega_R}$ can be written as:

$$\mathcal{I}_{\Omega_R}=\mathcal{I}_\Omega+\frac{\rho^3}{2}\langle \mathcal{S}'(u^R),u^R\rangle_{\Gamma_R}+o(\rho^3); \qquad (3.42)$$

and, in view of the asymptotic expansion (3.1), we finally have that the topological derivative satisfies the following identity:

$$\mathcal{T}_\Omega(\hat{x})=\frac{1}{2}\langle \mathcal{S}'(u^R),u^R\rangle_{\Gamma_R}. \qquad (3.43)$$

Proposition 2 establishes the differentiability property of the energy shape functional for this contact model with respect to the non-smooth perturbation denoted by $\mathcal{B}_\rho$. This is an abstract results, whose closed form for the topological derivative $\mathcal{T}_\Omega(\hat{x})$ is presented in the next section.

3.3. Topological derivative evaluation. As a main result from the previous section, we have that the energy shape functional admits an asymptotic expansion for $\rho\to 0^+$, see eqs.(3.1) and (3.43). This means that the asymptotic behavior of the energy in $C(R,\rho)$ holds in the whole domain Ω. Then, we only need to compute the topological derivative for the energy shape functional in $C(R,\rho)$, with its associated elastic problem (3.4). In order to evaluate the topological derivative, we can use the techniques available in the literature, see for instance [2, 21, 18, 24]. Finally, for an explicit and analytical formula for the topological derivative $\mathcal{T}_\Omega(\hat{x})$, we introduce the following result:

THEOREM 3. *The energy shape functional of an elastic solid, characterized by the constitutive equation* (2.5)*, with a spherical cavity of radius ρ with homogeneous Neumann boundary condition and centered at point $\hat{x}\in\Omega$, admits for $\rho\to 0^+$ the following asymptotic expansion:*

$$\mathcal{J}_{\Omega_\rho}(u_\rho)=\mathcal{J}_\Omega(u)+\rho^3\pi\mathbb{H}\sigma(u(\hat{x}))\cdot\varepsilon(u(\hat{x}))+o(\rho^3)\quad\forall\hat{x}\in\Omega, \qquad (3.44)$$

where $u(\hat{x})$ is the solution of the problem (2.7) evaluated at $\hat{x}$ and $\mathbb{H}$ is the fourth-order tensor defined as:

$$\mathbb{H}:=\frac{1-\upsilon}{7-5\upsilon}\left(10\mathbb{I}-\frac{1-5\upsilon}{1-2\upsilon}\mathrm{I}\otimes\mathrm{I}\right), \qquad (3.45)$$

where υ is the Poisson's ratio of the elastic medium, I *and* $\mathbb{I}$ *are the identities tensors of second- and fourth-order, respectively.*

Proof. The reader interested in the proof of this result may refer to [22, 13, 16]. □

REMARK 4. *The fourth-order tensor* $\mathbb{H}$ *in (3.44), can be interpreted as the polarization tensor associated to this problem. This is a important concept, since the topological derivative formula can be written explicitly in terms of this tensor. The reader interested in this topic may refer to the works [1, 5, 20].*

4. Final remarks. An analytical expression for the topological derivative of the energy shape functional associated to a frictionless contact model that allows a finite interpenetration between an elastic body and a rigid foundation, has been derived. We develop the asymptotic analysis for the case when a spherical void is introduced at an arbitrary point of the domain. By using the domain decomposition technique and the associated Steklov-Poincaré pseudodifferential operator, the differentiability of the energy was successfully established. The final formula is a general simple analytical expression in terms of the solution of the state equation and the constitutive parameters evaluated in each point of the unperturbed domain. From the asymptotic analysis, it was proved that the finite interpenetration condition on the potential contact zone does not contribute explicitly to the first order topological derivative. This means that the formula for the topological derivative is the same that for the classical elasticity problem for an isotropic and homogeneous medium. The contribution of the contact model in the topological derivative is through the displacement field, solution of the contact problem with the non-linear boundary condition (finite interpenetration). Finally, we remark that this information can be potentially used in the topological design of mechanical components, under contact conditions, to achieve a specified behavior.

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